MTH 166

Lecture-27

Directional Derivatives

Topic:

Vector Differential Calculus

Learning Outcomes:

1. To calculate directional derivatives

2. To find tangent plane to a surface

<u>Directional derivative of a scalar field</u>

Let $f(x, y, z)$ be a given scalar surface.

Let the given direction be $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

Then directional derivative of f in the direction of vector $\vec{b}$ is given by:

$$D_{\vec{b}}(f) = \vec{\nabla}f.\,\hat{b} = \vec{\nabla}f.\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right).\left(\frac{b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}\right)$$

$$= \frac{f_x b_1 + f_y b_2 + f_z b_3}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}$$

Note:

1. Maximum rate of increase (Minimum rate of decrease) $= \left|\vec{\nabla} f\right|$

It occurs in its own direction.

2. Minimum rate of increase (Maximum rate of decrease) $= -\left|\vec{\nabla} f\right|$

It occurs in its opposite direction.

Problem 1. Find the directional derivative of the scalar function $(x^2y - y^2z - xyz)$at the point $(1, -1, 0)$ in the direction $(\hat{\imath} - \hat{\jmath} + 2\hat{k})$.

Solution. Let $f = (x^2y - y^2z - xyz)$

$\Rightarrow f_x = (2xy - yz), f_y = (x^2 - 2yz - xz), f_z = (-y^2 - xy)$

$\vec{\nabla} f = (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}) = (2xy - yz)\hat{\imath} + (x^2 - 2yz - xz)\hat{\jmath} + (-y^2 - xy)\hat{k}$

At $(1, -1, 0)$: $\vec{\nabla} f = -2\hat{\imath} + \hat{\jmath} + 0\hat{k}$

Let $\vec{b} = (\hat{\imath} - \hat{\jmath} + 2\hat{k})$

$D_{\vec{b}}(f) = \vec{\nabla} f . \hat{b} = \vec{\nabla} f . \left(\frac{\vec{b}}{|\vec{b}|}\right)$

$\Rightarrow D_{\vec{b}}(f) = (-2\hat{\imath} + \hat{\jmath} + 0\hat{k}) . \frac{(\hat{\imath} - \hat{\jmath} + 2\hat{k})}{\sqrt{1+1+4}} = \frac{-2-1+0}{\sqrt{6}} = \frac{-3}{\sqrt{6}}$ **Answer.**

Problem 2. Find the directional derivative of the scalar function (xyz) at the point $(1,4,3)$ in the direction of line from $(1,2,3)$ to $(1,-1,-3)$.

Solution. Let $f = (xyz)$

$\Rightarrow f_x = (yz), f_y = (xz), f_z = (xy)$

$\vec{\nabla} f = \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right) = (yz)\hat{\imath} + (xz)\hat{\jmath} + (xy)\hat{k}$

At $(1,4,3)$: $\vec{\nabla} f = 12\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$

Let $\vec{b} = \Big((1-1)\hat{\imath} + (-1-2)\hat{\jmath} + (-3-3)\hat{k}\Big) = 0\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$

$$D_{\vec{b}}(f) = \vec{\nabla} f . \hat{b} = \vec{\nabla} f . \left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$\Rightarrow D_{\vec{b}}(f) = \left(12\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right).\frac{\left(0\hat{\imath}-3\hat{\jmath}-6\hat{k}\right)}{\sqrt{0+9+36}} = \frac{0-9-24}{\sqrt{45}} = \frac{-33}{3\sqrt{5}} = -\frac{11}{\sqrt{5}}$$ **Answer.**

Problem 3. Find a direction that gives the direction of maximum rate of increase of scalar function $(3x^2 + y^2 + 2z^2)$ at $(0,1,2)$. Find the maximum rate too.

Solution. Let $f = (3x^2 + y^2 + 2z^2)$

$\Rightarrow f_x = 6x, \quad f_y = 2y, \quad f_z = 4z$

$\vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}\right) = 6x\hat{\imath} + 2y\hat{\jmath} + 4z\hat{k}$

At (0,1,2): $\vec{\nabla} f = 0\hat{\imath} + 2\hat{\jmath} + 8\hat{k}$

which is the direction of maximum rate of increase.

Also, Maximum rate = $\left|\vec{\nabla} f\right| = \sqrt{0 + 4 + 64} = \sqrt{68} = 2\sqrt{17}$ **Answer.**

Problem 4. Find a direction that gives the direction of minimum rate of increase of scalar function $(x^3 - xy^2 + y^3)$ at $(-2,1)$. Find the minimum rate too.

Solution. Let $f = (x^3 - xy^2 + y^3)$

$\Rightarrow f_x = 3x^2 - 1, \quad f_y = 3y^2 - 2xy$

$\vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath}\right) = (3x^2 - 1)\hat{\imath} + (3y^2 - 2xy)\hat{\jmath}$

At $(-2,1)$: $\vec{\nabla} f = 11\hat{\imath} + 7\hat{\jmath}$

Direction of minimum rate of increase$= -\vec{\nabla} f = -(11\hat{\imath} + 7\hat{\jmath})$

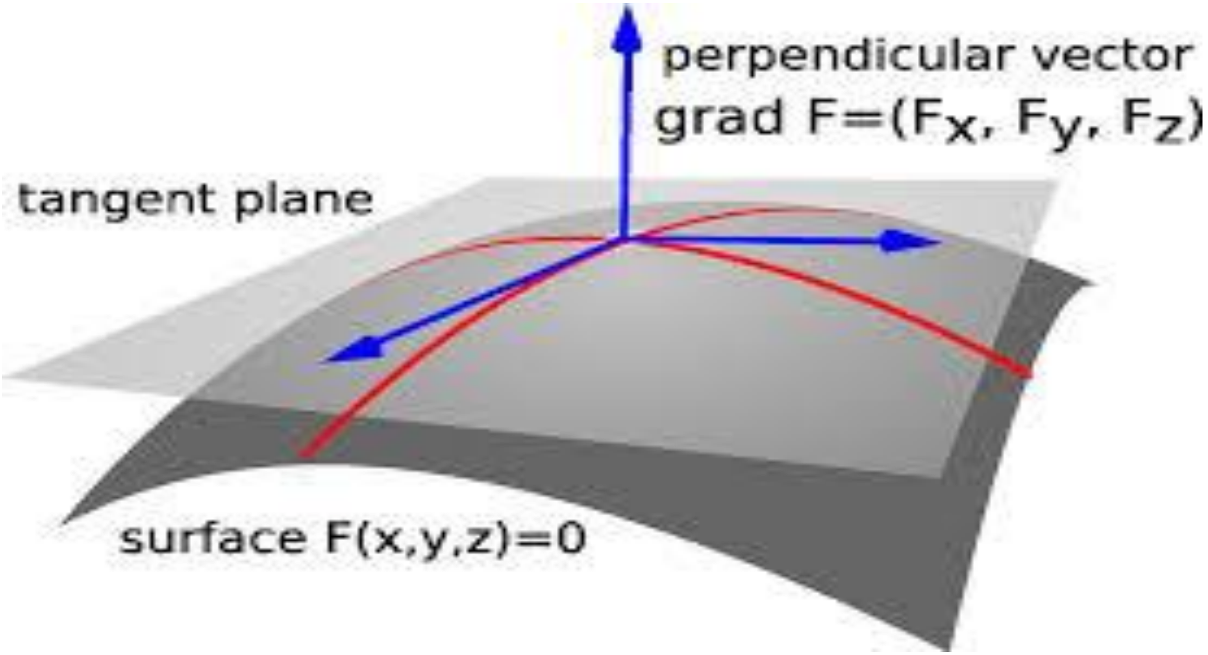
Also, Minimum rate $= -\left|\vec{\nabla} f\right| = -\sqrt{121 + 49} = -\sqrt{170}$ **Answer.**

Tangent Plane to a Scalar Surface

Let $f(x, y, z)$ be a given scalar surface.

Then equation of tangent plane to the surface $f(x, y, z)$ at the point $P_0(x_0, y_0, z_0)$ is:

$$(x - x_0)f_x(P_0) + (y - y_0)f_y(P_0) + (z - z_0)f_z(P_0) = 0$$



Problem. Find equation of tangent plane to surface $(x^2 - 3y^2 - z^2 = 2)$ at (3,1,2).

Solution. Let $f = x^2 - 3y^2 - z^2 = 2$

$\Rightarrow f_x = 2x, \quad f_y = -6y, \quad f_z = -2z$

At (3,1,2): $f_x = 6, \quad f_y = -6, \quad f_z = -4$

Equation of tangent plane is given by:

$(x - x_0)f_x(P_0) + (y - y_0)f_y(P_0) + (z - z_0)f_z(P_0) = 0$

$\Rightarrow (x - 3)(6) + (y - 1)(-6) + (z - 2)(-4) = 0$

$\Rightarrow 6x - 18 - 6y + 6 - 4z + 8 = 0$

$\Rightarrow 6x - 6y - 4z = 4$

$\Rightarrow 3x - 3y - 2z = 2$ **Answer.**